Dispersionless scalar integrable hierarchies, Whitham hierarchy and the quasi-classical $\bar{\partial}$-dressing method.*

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Abstract

The quasi-classical limit of the scalar nonlocal $\bar{\partial}$-problem is derived and a quasi-classical version of the $\bar{\partial}$-dressing method is presented. Dispersionless KP, mKP and 2DTL hierarchies are discussed as illustrative examples. It is shown that the universal Whitham hierarchy it is nothing but the ring of symmetries for the quasi-classical $\bar{\partial}$-problem. The reduction problem is discussed and, in particular, the d2DTL equation of B type is derived.

1 Introduction

A considerable interest has been paid recently to dispersionless or quasi-classical limits of integrable equations and hierarchies (see *e.g.* [1]-[13] and references therein). Study of dispersionless hierarchies is of great importance since they arise in the analysis of various problems in physics, mathematics and applied mathematics from the theory of quantum fields and strings [14]-[16] to the theory of conformal maps on the complex plane [17]-[18].

Different methods have been used to study dispersionless equations and hierarchies [1]-[13]. In particular, several 1+1-dimensional equations and systems have been analyzed by the quasi-classical version of the inverse scattering transform, including the local Riemann-Hilbert problem approach [2],[3],[11]-[13],[19].

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Similar study of the 2+1-dimensional equations and hierarchies, like KP and 2DTL, has been missing. Recently this problem has been addressed in [20] and the quasi-classical $\bar{\partial}$-dressing approach to the dispersionless KP hierarchy has been proposed.

In this paper we consider a class of scalar dispersionless integrable hierarchies governed by the scalar $\bar{\partial}$-problem with the dKP, mdKP and d2DTL hierarchies as particular cases. We derived the general form of the quasi-classical $\bar{\partial}$-problem. It is given by the system

$$\frac{\partial S}{\partial \bar{\lambda}} = W\left(\lambda, \bar{\lambda}; \frac{\partial S}{\partial \lambda}\right), \tag{1.1}$$

$$\frac{\partial \varphi}{\partial \bar{\lambda}} = W'\left(\lambda, \bar{\lambda}; \frac{\partial S}{\partial \lambda}\right)\frac{\partial \varphi}{\partial \lambda} + \widetilde{W}\left(\lambda, \bar{\lambda}; \frac{\partial S}{\partial \lambda}\right)\frac{\partial^2 S}{\partial \lambda^2}\,\varphi, \tag{1.2}$$

for $\lambda \in G$, where G is a domain in the complex plane $\mathbb{C}$ and W and $\widetilde{W}$ are some functions. The type of hierarchy is specified by the undressed part $S_0(\lambda, T)$ of S and the domain G. A quasiclassical $\bar{\partial}$-dressing method based on the system (1.1)-(1.2) allows us to construct dispersionless integrable hierarchies and provides us a method for finding their solutions. The dKP, dmKP and d2DTL hierarchies are considered as illustrative examples.

Symmetries of the quasi-classical $\bar{\partial}$-problem (1.1)-(1.2) are defined by linear Beltrami-type equations and form an infinite-dimensional ring. It is shown that this ring, parametrized by symmetry parameters, is nothing but the universal Whitham hierarchy introduced in [8]. In particular, the dKP, mdKP and m2DTL hierarchies are special subrings of symmetries for problems (1.1)-(1.2).

We discuss also the reduction of the dispersionless hierarchies and present the dispersionless 2DTL equation of B type.

Equations of the form (1.1)-(1.2) are well-known in the complex-analysis; in particular, in connection with quasi-conformal mappings in the plane (see *e.g.* [21]-[23]). Thus, there is a close connection between the theory of quasi-classical integrable hierarchies and the theory of quasi-conformal mappings.

2 Dispersionless hierarchies and universal Whitham hierarchy

We begin by reminding some relevant formulas for dispersionless hierarchies and choose the Kadomtsev-Petviashili (KP) hierarchy to illustrate their main features. The usual KP hierarchy is an infinite set of the compatibility condition for the system

$$L\psi = \lambda\psi \tag{2.1}$$

$$\frac{\partial \psi}{\partial t_n} = (L^n)_+\,\psi \tag{2.2}$$

where $L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + ..., \partial = \frac{\partial}{\partial t}, (L^n)_+$ denotes the pure differential part of the operator L^n, λ is a spectral parameter and ψ is a common KP

wave-function. The KP equation itself is the equation for coefficient u_1 as a function of the first three times t_1, t_2, t_3. For the modified KP (mKP) hierarchy the operator L is of the form $L = \partial + u_0 + u_1\partial^{-1} + u_2\partial^{-2} + \ldots$ while for the two-dimensional Toda lattice (2DTL) hierarchy one needs two operators L_1 and L_2 [10].

The dispersionless KP (dKP) hierarchy is a formal limit $\varepsilon \to 0$ of the KP hierarchy for which [1]-[10]

$$u_k\left(\frac{T_n}{\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} u_{k_0}(T) + O(\varepsilon), \tag{2.3}$$

and

$$\psi\left(\frac{T_n}{\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} e^{\frac{1}{\varepsilon}S(\lambda,T)+O(\varepsilon)}, \tag{2.4}$$

where T_n are slow times.

Under such a limit, equation (2.1) gives rise to the Laurent series $\mathcal{L}= p + \sum_{n=1}^{\infty} u_n(T)\, p^{-n}$, where $p = \frac{\partial S}{\partial T_1}$ while equations (2.2) become

$$\frac{\partial p}{\partial T_n} = \frac{\partial B_n(p)}{\partial T_1} \tag{2.5}$$

where $B_n(p) = [\mathcal{L}^n(p)]_+$ and $[\mathcal{L}^n]_+$ denotes here a polynomial part of $\mathcal{L}^n$. The compatibility conditions for (2.5) are given by the infinite set of equation

$$\frac{\partial B_n}{\partial T_m} - \frac{\partial B_n}{\partial T_m} + \{B_n, B_m\} = 0 \tag{2.6}$$

where the Poisson bracket $\{,\}$ is defined as

$$\{f,g\} = \frac{\partial f}{\partial p}\,\frac{\partial g}{\partial T_1} - \frac{\partial f}{\partial T_1}\,\frac{\partial g}{\partial p} \qquad . \tag{2.7}$$

Equation (2.5) or (2.6) represent the dKP hierarchy. Similarly, the dmKP hierarchy is given by equations of the form (2.5)-(2.7) with $\mathcal{L}= p + \sum_{n=1}^{\infty} u_n(T)\, p^{-n}$ [25]. The d2DTL hierarchy can be written by a set of equations similar to (2.5)-(2.7) for two Laurent series $\mathcal{L}_1$ and $\mathcal{L}_2$ [10] with the substitution $p \to e^p$.

A more general dispersionless-like hierarchy has been introduced in [8]. This universal Whitham hierarchy is given by the infinite set of equations

$$\frac{\partial \Omega_A}{\partial T_B} - \frac{\partial \Omega_B}{\partial T_A} + \{\Omega_A, \Omega_B\} = 0 \qquad , \qquad A, B = 1, 2, 3, \ldots \tag{2.8}$$

where $\Omega_A(p,T)$ are arbitrary holomorphic functions of p. As it has been shown in [8], the dKP, d2DTL and Benney hierarchies are particular cases.

3 Quasi-classical $\bar{\partial}$-problem

The $\bar{\partial}$-dressing method is a powerful tool to study usual integrable equations and hierarchies [25]-[27]. In this paper we shall formulate its quasi-classical version. We shall demonstrate that it provides an effective method to construct and study dispersionless hierarchies.

We begin with the derivation of the quasi-classical limit of the basic $\bar{\partial}$-problem.

The usual scalar integrable hierarchies are associated with the following scalar linear nonlocal problem (see [25]-[27])

$$\frac{\partial\,\chi(\lambda,\overline{\lambda};t)}{\partial\,\overline{\lambda}} = \int\int_{\mathbb{C}} d\mu\wedge d\overline{\mu}\;\chi(\mu,\overline{\mu};t)\;g(\mu,t)\,R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})\,g^{-1}(\lambda,t) \tag{3.1}$$

where λ is a complex variable ("spectral parameter"), $\overline{\lambda}$ denotes complex conjugation of $\lambda, \chi(\lambda,\overline{\lambda};\mu)$ is a complex-valued function on the complex plane $\mathbb{C}$ ($\lambda,\overline{\lambda}\in\mathbb{C}$), the kernel $R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})$ is the $\bar{\partial}$-data. Usually, it is assumed that the function χ has a canonical normalization (*i.e.*

$$\chi\to 1+\frac{\chi_1}{\lambda}+\frac{\chi_2}{\lambda^2}+...,\quad \lambda\to\infty,$$

and that the problem (3.1) is uniquely solvable. Concrete integrable hierarchies are specialized by the form of the function $g(\lambda,t)=\exp(S_0(\lambda,t))$ and by the domain G of the support for the $\bar{\partial}$-data R_0 ($R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})=0$ for $\mu,\lambda\in\mathbb{C}/G$). For the KP hierarchy $S_0=\sum_{k=1}^{\infty}\lambda^k t_k$ and G ia a disk with center at the origin, while for the 2DTL hierarchy $S_0(\lambda;x,y,n)=n\ln\lambda+\sum_{k=1}^{\infty}\lambda^k x_k+\sum_{k=1}^{\infty}\lambda^{-k}y_k$ where x_k and y_k are continuous variables and n is an integer discrete variable. The domain G in this case is an annulus $a\leq|\lambda|\leq b$. Given $g(\lambda)$ the $\bar{\partial}$-dressing method provides us with the corresponding hierarchy of nonlinear equations and their linear problems [25]-[27]. Solutions of nonlinear equations are given by the function χ evaluated at certain points λ_0. For instance, for the KP hierarchy $u_1=-2\frac{\partial\chi_1(t)}{\partial t_1}$.

In order to derive the quasi-classical limit of the $\bar{\partial}$-problem (3.1) we first introduce slow variables T ($t_i=\frac{T_i}{\varepsilon}$ for KP and mKP, $x_i=\frac{X_i}{\varepsilon}$, $y_i=\frac{Y_i}{\varepsilon}$, $n=\frac{T}{\varepsilon}$ for 2DTL) for small ε and proceed to the limit $\varepsilon\to 0$. In this limit $g\left(\frac{T}{\varepsilon}\right)=\exp\left[\frac{S_0(\lambda,T)}{\varepsilon}\right]$. Motivated by the formula of the type (2.4) and by the structure of equation (3.1) we will look for solutions χ of the form

$$\chi\left(\lambda,\overline{\lambda};\frac{T}{\varepsilon}\right)=\widehat{\chi}(\lambda,\overline{\lambda};T;\varepsilon)\;e^{\frac{\widetilde{S}(\lambda,\overline{\lambda};T)}{\varepsilon}} \tag{3.2}$$

where $\widetilde{S}(\lambda,\overline{\lambda};T)$ is a certain function and

$$\widehat{\chi}(\lambda,\overline{\lambda};T;\varepsilon)=\sum_{n=0}^{\infty}\widehat{\chi}_n(\lambda,\overline{\lambda};T)\,\varepsilon^n \quad . \tag{3.3}$$

It is clear that only for special $\bar{\partial}$-data R_0 equation (3.1) will have a well-defined limit as $\varepsilon \to 0$. Thus is not difficult to see that the $\bar{\partial}$-data of the form

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda};\varepsilon) \,=\sum_{k=0}^{\infty}\Gamma_k(\mu,\overline{\mu})\,\varepsilon^{k-1}\,\delta^{(k)}(\mu-\lambda-\varepsilon\,\alpha_k(\lambda,\overline{\lambda})) \tag{3.4}$$

do a job. Here $\Gamma_k(\mu,\overline{\mu})$, $\alpha_k(\lambda,\overline{\lambda})$ are arbitrary functions ($\Gamma_k=0$ at $\lambda\in\mathbb{C}/G$) and $\delta^{(k)}$ is the k-derivative Dirac delta-function. Indeed, substituting (3.4) into (3.1), one gets

$$\frac{\partial\widehat{\chi}(\lambda,\overline{\lambda};T,\epsilon)}{\partial\overline{\lambda}}+\frac{1}{\varepsilon}\frac{\partial S(\lambda,\overline{\lambda};T)}{\partial\overline{\lambda}}\;\widehat{\chi}(\lambda,\overline{\lambda},T,\epsilon)=$$

$$=\;\int\!\!\int_{\mathbb{C}} d\mu\wedge d\overline{\mu}\;\widehat{\chi}(\mu,\overline{\mu},\epsilon)\;e^{\frac{S(\mu,T)-S(\lambda,T)}{\varepsilon}}\;\sum_{k=0}^{\infty}\Gamma_k(\mu,\overline{\mu})\,\varepsilon^{k-1}\delta^{(k)}(\mu-\lambda-\varepsilon\alpha_k(\lambda,\overline{\lambda})\,) \tag{3.5}$$

where

$$S(\lambda,\overline{\lambda};T):=\widetilde{S}(\lambda,\overline{\lambda};T)+S_0(\lambda;T).$$

Evaluating in (3.5) the terms of the order $\frac{1}{\varepsilon}$, one obtains

$$\frac{\partial S(\lambda,\overline{\lambda};T)}{\partial\overline{\lambda}}=W\left(\lambda,\overline{\lambda};\frac{\partial S}{\partial\lambda}\right) \tag{3.6}$$

where

$$W\left(\lambda,\overline{\lambda};\frac{\partial S}{\partial\lambda}\right)=\sum_{k=0}^{\infty}(-1)^k\,\Gamma_k(\lambda,\overline{\lambda})\,\left(\frac{\partial S}{\partial\lambda}\right)^k\;e^{\alpha_k(\lambda,\overline{\lambda})\,\frac{\partial S}{\partial\lambda}}\qquad. \tag{3.7}$$

Furthermore, the terms of zero order in ε in (3.5) give (the contribution proportional to $\widehat{\chi}_1$ disappears due to (3.6)):

$$\frac{\partial\varphi}{\partial\overline{\lambda}}=W'\left(\lambda,\overline{\lambda};\frac{\partial S}{\partial\lambda}\right)\,\frac{\partial\varphi}{\partial\lambda}+\widetilde{W}\left(\lambda,\overline{\lambda};\frac{\partial S}{\partial\lambda}\right)\,\frac{\partial^2 S}{\partial\lambda^2}\varphi \tag{3.8}$$

where

$$\varphi:=\widehat{\chi}_0,\quad W'(\lambda,\overline{\lambda};\xi)\,:=\frac{\partial W(\lambda,\overline{\lambda};\xi)}{\partial\xi},$$

and

$$\begin{aligned}\widetilde{W}\;&:=\;(-1)\,\Gamma_1\,e^{\alpha_1\frac{\partial S}{\partial\lambda}}\frac{1}{2}\alpha_1^2\frac{\partial S}{\partial\lambda}+(-1)^2\,\Gamma_2\,e^{\alpha_2\frac{\partial S}{\partial\lambda}}\left[1+\frac{1}{2}\alpha_2^2\left(\frac{\partial S}{\partial\lambda}\right)^2\right]+\\&+(-1)^3\,\Gamma_3\,e^{\alpha_3\frac{\partial S}{\partial\lambda}}\left[3\frac{\partial S}{\partial\lambda}+\frac{1}{2}\alpha_3^2\left(\frac{\partial S}{\partial\lambda}\right)^3\right]+\ldots\end{aligned} \tag{3.9}$$

Since $\frac{\partial S_0}{\partial \overline{\lambda}} = 0$ at $\lambda \in G$, then the equations (3.6)-(3.9) for $\lambda \in G$ can be rewritten as

$$\begin{aligned} \frac{\partial S}{\partial \overline{\lambda}} &= W\left(\lambda, \overline{\lambda}; \frac{\partial S}{\partial \lambda}\right), \qquad (3.10) \\ \frac{\partial \varphi}{\partial \overline{\lambda}} &= W' \frac{\partial \varphi}{\partial \lambda} + \widetilde{W} \frac{\partial^2 S}{\partial \lambda^2}\, \varphi. \qquad (3.11) \end{aligned}$$

Equations (3.10) and (3.11) are the quasi-classical limit of the nonlocal $\bar{\partial}$-problem (3.1). The derivation given above suggests that the quasiclassical limit of the $\bar{\partial}$-problem (3.1) is given by equations (3.10), (3.11) also for a more general than (3.4) $\bar{\partial}$-data R_0.

The function S is widely used in the analysis of the dispersionless limits of the integrable hierarchies [4]-[10]. Within the $\bar{\partial}$-approach it is a nonholomorphic function of the "spectral" variable λ and obeys the nonlinear $\bar{\partial}$-equation (3.10) (for $\lambda \in G$). The function $\varphi = \widehat{\chi}_0$ obeys the local $\bar{\partial}$-problem (3.11) of the Beltrami type. Note that the ratio ϕ of two solutions φ_1 and φ_2 of equation (3.11) satisfies the Beltrami equation

$$\frac{\partial \phi}{\partial \overline{\lambda}} = W'\left(\lambda, \overline{\lambda}; \frac{\partial S}{\partial \lambda}\right) \frac{\partial \phi}{\partial \lambda}. \qquad (3.12)$$

For the Orlov's function $M = \frac{\partial S}{\partial \lambda}$ equations (3.10) and (3.11) take the form of quasi-linear equations

$$\begin{aligned} \frac{\partial M}{\partial \overline{\lambda}} &= \frac{\partial}{\partial \lambda} W\left(\lambda, \overline{\lambda}; M\right), \qquad (3.13) \\ \frac{\partial \varphi}{\partial \overline{\lambda}} &= W'\left(\lambda, \overline{\lambda}; M\right) \frac{\partial \varphi}{\partial \lambda} + \widetilde{W}\left(\lambda, \overline{\lambda}; M\right) \frac{\partial M}{\partial \lambda}\, \varphi. \qquad (3.14) \end{aligned}$$

In the particular case of the $\bar{\partial}$-data R_0 given by (3.4) with all $\alpha_k \equiv 0$ one has $\widetilde{W} = \frac{1}{2} W''\left(\lambda, \overline{\lambda}; \frac{\partial S}{\partial \lambda}\right)$.

Quasi-classical $\bar{\partial}$-problems (3.10), (3.11) are basic equations for our approach. The equations of the type (3.10), (3.11) are well known and widely studied in the theory of nonlinear elliptic systems with two independent variables and in complex analysis (see *e.g.* [28],[21]-[23]). One theorem from the theory of such equations will be crucial for our further constructions. This theorem (see theorem 3.32 from [28]) states that , under certain mild condition on A(see the appendix), the only solution of the Beltrami equation $\frac{\partial Z}{\partial \overline{\lambda}} = A \frac{\partial Z}{\partial \lambda}$ in $\mathbb{C}$ which vanish as $\lambda \to \infty$ is $Z \equiv 0$.

4 Quasi-classical $\bar{\partial}$-dressing method

The principal goal of the $\bar{\partial}$-dressing method based on equations (3.10), (3.11) is the same as of the original $\bar{\partial}$-dressing method [25]-[27]. It is to extract the nonlinear differential equations from the quasi-classical $\bar{\partial}$-problems.

Now the time dependence of the functions S and φ is encoded in the undressed functions $S_0(\lambda, T)$. Since $\widetilde{S} = 1 + \frac{S_1}{\lambda} + \frac{S_2}{\lambda^2} + ...$ at $\lambda \to \infty$ then the behavior of $\frac{\partial S}{\partial T_A}$ for large λ is completely defined by

$$\frac{\partial S}{\partial T_A} = \frac{\partial S_0}{\partial T_A} + \frac{1}{\lambda}\frac{\partial S_1}{\partial T_A} + ... \tag{4.1}$$

where T_A is a slow time.

A basic property of the nonlinear equation (3.10) is that it implies the linear Beltrami equation for the infinitesimal variations δS (symmetries):

$$\frac{\partial}{\partial \overline{\lambda}}(\delta S) = W'\left(\lambda, \overline{\lambda}; \frac{\partial S}{\partial \lambda}\right) \frac{\partial}{\partial \lambda}(\delta S) \qquad . \tag{4.2}$$

In particular, for any time T_A

$$\frac{\partial}{\partial \overline{\lambda}}\left(\frac{\partial S}{\partial T_A}\right) = W'\left(\lambda, \overline{\lambda}; \frac{\partial S}{\partial \lambda}\right) \frac{\partial}{\partial \lambda}\left(\frac{\partial S}{\partial T_A}\right) \qquad . \tag{4.3}$$

Any power of solution of the Beltrami equation is a solution too as well as any differentiable function of two solutions. So together with $\frac{\partial S}{\partial T_{A_1}}, ..., \frac{\partial S}{\partial T_{A_n}}$ any differentiable function $f\left(\frac{\partial S}{\partial T_{A_1}}, ..., \frac{\partial S}{\partial T_{A_n}}\right)$ with arbitrary n is a solution of equation (4.3). Thus the symmetries of the problem (3.10) form a ring.

Due to (4.1), the functions $f\left(\frac{\partial S}{\partial T_{A_1}}, ..., \frac{\partial S}{\partial T_{A_n}}\right)$ have singularities in certain points. The functions $f_0\left(\frac{\partial S}{\partial T_{A_1}}, ..., \frac{\partial S}{\partial T_{A_n}}\right)$ which are bounded in $\mathbb{C}$ and vanish as $\lambda \to \infty$ are very special. According to the Vekua's theorem mentioned in the end of the previous section they vanish identically. So we have the nonlinear equations

$$f_0\left(\frac{\partial S}{\partial T_{A_1}}, ..., \frac{\partial S}{\partial T_{A_n}}\right) = 0 \qquad . \tag{4.4}$$

Note that in contrast to the usual hierarchies we get nonlinear equations for the "wave function" S (the classical action).

Now we turn to the $\bar{\partial}$-problem (3.11). It is linear one. So the construction is similar to that of usual case. Namely, suppose that one has found a solution φ_0 to equation (3.11) of the form $\varphi_0 = L\varphi$ where L is a certain linear operator and φ_0 is bounded and vanishes as $\lambda \to \infty$. Since $\varphi \to 1 + \frac{\varphi_1}{\lambda} + ...$ as $\lambda \to \infty$ the ratio $\frac{\varphi_0}{\varphi}$ vanishes as $\lambda \to \infty$ and obeys the Beltrami equation

$$\frac{\partial}{\partial \overline{\lambda}}\left(\frac{\varphi_0}{\varphi}\right) = W' \frac{\partial}{\partial \lambda}\left(\frac{\varphi_0}{\varphi}\right) \qquad . \tag{4.5}$$

Then according to the Vekua's theorem $\frac{\varphi_0}{\varphi}$ vanishes identically and, consequently,

$$L\varphi = 0 \tag{4.6}$$

that is the desired linear problem for the wavefunction φ. Note that one can get the same results assuming that the problem (3.11) with canonically normalized φ is uniquely solvable.

Equations (4.4) and (4.6) are the basic equations associated with the quasi-classical $\bar{\partial}$-problems (3.10), (3.11). They are compatible by construction. Equation (4.4), (4.6) provide us also with equations for functions $u_k(T)$ which depend only on the times T. Usually one has infinite families of equations of the type (4.4), (4.6). So the quasi-classical $\bar{\partial}$-problems (3.10), (3.11) give rise to an infinite hierarchy of integrable quasi-classical (or dispersionless) equations.

5 dKP and dmKP hierarchies

Let us consider concrete examples to illustrate the general scheme. We start with the dKP hierarchy. In this case $S_0(\lambda,T)=\sum_{n=1}^{\infty}\lambda^n T_n$ and $\frac{\partial S}{\partial T_n}=\lambda^n+\frac{1}{\lambda}\frac{\partial S_1}{\partial T_n}+\frac{1}{\lambda^2}\frac{\partial S_2}{\partial T_n}+...\ (n=1,2,....)$. Since $\frac{\partial S}{\partial T_n}$ have power singularities at infinity the desired function f_0 will be, clearly, polynomials. Taking, for instance, the derivatives $\frac{\partial S}{\partial T_2}$ and $\frac{\partial S}{\partial T}$ we readily see that the difference $\frac{\partial S}{\partial T_2}-\left(\frac{\partial S}{\partial T_1}\right)^2$ behaves as $-2\frac{\partial S_1}{\partial T_1}+O(\frac{1}{\lambda})$ as $\lambda\to\infty$. Thus, the desired function f_0 is $\frac{\partial S}{\partial T_2}-\left(\frac{\partial S}{\partial T_1}\right)^2+2\frac{\partial S_1}{\partial T_1}$. So we get the equation

$$\frac{\partial S}{\partial T_2}-\left(\frac{\partial S}{\partial T_1}\right)^2-u=0 \tag{5.1}$$

where $u=-2\frac{\partial S_1}{\partial T_1}$.

Analogously, taking the derivatives $\frac{\partial S}{\partial T_3}$ and $\frac{\partial S}{\partial T_1}$ one easily concludes that the combination $\frac{\partial S}{\partial T_3}-\left(\frac{\partial S}{\partial T_1}\right)^3-V_1\frac{\partial S}{\partial T_1}-V_0$ vanishes at $\lambda\to 0$ if $V_1=-3\frac{\partial S}{\partial T_1}$ and $V_0=-3\frac{\partial S_2}{\partial T_1}=-\frac{3}{2}\frac{\partial S_1}{\partial T_2}$. So one gets the other function f_0 and the equation

$$\frac{\partial S}{\partial T_3}-\left(\frac{\partial S}{\partial T_1}\right)^3-V_1\frac{\partial S}{\partial T_1}-V_0=0 \tag{5.2}$$

where $\frac{\partial V_0}{\partial T_1}=\frac{3}{4}\frac{\partial u}{\partial T_2}$.

In a similar manner one constructs an infinite family of equations

$$\frac{\partial S}{\partial T_n}-\left(\frac{\partial S}{\partial T_1}\right)^n-\sum_{k=0}^{n-2}V_{nk}(T)\ \left(\frac{\partial S}{\partial T_1}\right)^k=0 \qquad , \quad n=1,2,3,... \tag{5.3}$$

with appropriate coefficients $V_{nk}(T)$.

Equations (5.3) are nothing but the equations (2.5) of the dKP hierarchy. Evaluating the left-hand-sides of (5.3) at $\lambda\to\infty$, taking the term of the order $\frac{1}{\lambda}$ and using other equations (5.3), one gets the dKP hierarchy for the function u and, in particular, the dKP equation

$$u_{T_1T_3}=\frac{3}{2}(u\,u_{T_1})_{T_1}+\frac{3}{4}u_{T_2T_2} \qquad . \tag{5.4}$$

In a usual manner equation (5.4) arises as the compatibility conditions for equations (5.1) and (5.2). Equations (5.3) implies the hierarchy of nonlinear equations for the function S only. Indeed, eliminating all coefficients $u_{nk}(T)$ from (5.3) one gets the family of equations

$$\frac{\partial S}{\partial T_n} = F_n\left(\frac{\partial S}{\partial T_1}, \frac{\partial S}{\partial T_2}\right) \qquad , \qquad n = 3, 4, 5, ...$$

The lowest of these equations is of the form

$$\frac{\partial^2 S}{\partial T_1 \partial T_3} = \frac{3}{4}\frac{\partial^2 S}{\partial T_2^2} + \frac{3}{2}\left[\frac{\partial S}{\partial T_2} - \left(\frac{\partial S}{\partial T_1}\right)^2\right] \frac{\partial^2 S}{\partial T_1^2} \qquad .$$

Now we proceed to the $\bar{\partial}$-problem (3.11). For simplicity we restrict ourselves to the case $\widetilde{W} = \frac{1}{2}W''$. It is not difficult to show, differentiating (3.11) and using equations (5.1)-(5.2), that the function $Z = L\varphi = \frac{\partial\varphi}{\partial T_2} - 2\frac{\partial S}{\partial T_1}\frac{\partial\varphi}{\partial T_1} - \frac{\partial^2 S}{\partial T_1^2}\varphi + \widetilde{u}\varphi$ where $\widetilde{u} = -2\frac{\partial\varphi_1}{\partial T_1}$ obeys the equation

$$\frac{\partial Z}{\partial\overline{\lambda}} = W'\frac{\partial Z}{\partial\lambda} + \frac{1}{2}W''\frac{\partial^2 S}{\partial\lambda^2}Z \tag{5.5}$$

and vanish at $\lambda \to 0$. Consequently the ratio $\frac{Z}{\varphi}$ obeys the Beltrami equation $\frac{\partial}{\partial\overline{\lambda}}\left(\frac{Z}{\varphi}\right) = W'\frac{\partial}{\partial\lambda}\left(\frac{Z}{\varphi}\right)$ and $\frac{Z}{\varphi} = O(\frac{1}{\lambda})$ as $\lambda \to \infty$. According to the Vekua's theorem this ratio vanishes identically and, consequently, we get the linear problem $Z = 0$, *i.e.*

$$\frac{\partial\varphi}{\partial T_2} - 2\frac{\partial S}{\partial T_1}\frac{\partial\varphi}{\partial T_1} - \frac{\partial^2 S}{\partial T_1^2}\varphi + \widetilde{u}\varphi = 0 \qquad . \tag{5.6}$$

In a similar manner, one gets the equation

$$\frac{\partial\varphi}{\partial T_3} - 6\left[2\left(\frac{\partial S}{\partial T_1}\right)^2 + \widetilde{u}\right]\frac{\partial\varphi}{\partial T_1} - 3\left[4\frac{\partial S}{\partial T_1}\frac{\partial^2 S}{\partial T_1^2} + \frac{\partial\widetilde{u}}{\partial T_1} - 2\widetilde{u}\frac{\partial S}{\partial T_1} + \widetilde{w}\right]\varphi \quad = \quad 0 \tag{5.7}$$

and higher-time equations

$$\frac{\partial\varphi}{\partial T_n} - A_n\frac{\partial\varphi}{\partial T_1} - B_n\varphi = 0 \qquad , \qquad n = 1, 2, 3... \tag{5.8}$$

All linear problems (5.6)-(5.8) are compatible by construction.

In the particular case $\widetilde{u} = \widetilde{w} = 0$ equation (5.6) is known as the transport equation within the quasiclassical approximation in quantum mechanics (see *e.g.* [29]). Note that in the case $\widetilde{u} = \widetilde{w} = 0$ equations (5.6), (5.7) (and also (5.8)) take the form of conservation laws

$$\begin{aligned} &\tfrac{\partial\phi}{\partial T_2} - \tfrac{\partial}{\partial T_1}\left(\tfrac{\partial S}{\partial T_1}\phi\right) = 0 \qquad , \\ &\tfrac{\partial\phi}{\partial T_3} - \tfrac{\partial}{\partial T_1}\left(12\left(\tfrac{\partial S}{\partial T_1}\right)^2\phi\right) = 0 \qquad . \end{aligned} \tag{5.9}$$

Considering the adjoint dKP hierarchy for which $\psi^*(T) = \widetilde{\chi}^*(T,\lambda;\varepsilon)e^{-\frac{S}{\varepsilon}}$, one gets the same equation (4.1) for S and equations for φ^* which are adjoint to (5.6)-(5.7). It is interesting that the quantity $\varphi(\lambda,T)\,\varphi^*(\lambda,T)$ obeys exactly equations (5.9).

Our second example is given by the dmKP hierarchy. In this case $S_0 = \sum_{k=1}^{\infty}\lambda^{-k}T_k$ and $\frac{\partial S}{\partial T_k} = \frac{1}{\lambda^k} + \frac{\partial \widetilde{S}(\lambda,T)}{\partial T_k}$ where $\widetilde{S}(\lambda,T)$ is holomorphic around $\lambda = 0$. So to construct required functions f_0 one has to cancel singularities around $\lambda = 0$. Taking again the derivatives $\frac{\partial S}{\partial T_1}$ and $\frac{\partial S}{\partial T_2}$ one readily see that the combination $\frac{\partial S}{\partial T_2} - \left(\frac{\partial S}{\partial T_1}\right)^2$ has only simple pole at $\lambda = 0$. To cancel it, we subtract $V(T)\,\frac{\partial S}{\partial T_1}$ where $V(T) = -2\frac{\partial \widetilde{S}(\lambda=0)}{\partial T_2}$. Then at $\lambda = 0$ one has $\frac{\partial S}{\partial T_2} - \left(\frac{\partial S}{\partial T_1}\right)^2 - V(T)\,\frac{\partial S}{\partial T_1} = O\left(\frac{1}{\lambda}\right)$. So due to the Vekua's theorem we conclude

$$\frac{\partial S}{\partial T_2} - \left(\frac{\partial S}{\partial T_1}\right)^2 - V(T)\,\frac{\partial S}{\partial T_1} = 0 \tag{5.10}$$

where $V(T) = -2\frac{\partial \widetilde{S}(\lambda=0,T)}{\partial T_2}$. Taking the derivatives $\frac{\partial S}{\partial T_1}$ and $\frac{\partial S}{\partial T_3}$, one finds the equation

$$\frac{\partial S}{\partial T_3} - \left(\frac{\partial S}{\partial T_1}\right)^3 - \frac{3}{2}V\ \left(\frac{\partial S}{\partial T_1}\right)^2 - \left(\frac{3}{4}V^2 - 3W\right)\frac{\partial S}{\partial T_1} = 0 \tag{5.11}$$

where $\widetilde{S}(\lambda) = \widetilde{S}(0) + \lambda W(T) + ...$ as $\lambda \to 0$. Analogously, one obtain the infinite hierarchy of equations

$$\frac{\partial S}{\partial T_n} - \left(\frac{\partial S}{\partial T_1}\right)^n - \sum_{k=1}^{n-1} V_{nk}(T)\ \left(\frac{\partial S}{\partial T_1}\right)^k = 0 \qquad , \qquad n = 1,2,3... \tag{5.12}$$

Equations (5.12) give us the dmKP hierarchy (see *e.g.* [24]). The simplest of these equations is the dmKP equation

$$V_t + \frac{3}{2}V^2V_x - \frac{3}{4}V_x\partial_x^{-1}V_y - \frac{3}{4}\partial_x^{-1}V_{yy} = 0 \qquad . \tag{5.13}$$

Analogously to the KP case, one can construct also the hierarchy of linear problems for the function φ. The simplest of them is of the form

$$\frac{\partial \varphi}{\partial T_2} - \left(2\frac{\partial S}{\partial T_1} + V\right)\frac{\partial \varphi}{\partial T_1} - \frac{\partial^2 S}{\partial T_1^2}\,\varphi = 0 \qquad . \tag{5.14}$$

Analogously to the dKP case equations (5.12) imply the hierarchy of equations for S only.

6 Dispersionless two-dimensional Toda lattice (2DTL) hierarchy

Our third example is the d2DTL hierarchy. In this case $S(\lambda;X,Y,T) = T\ln\lambda + \sum_{n=1}^{\infty}\lambda^n X_n + \sum_{n=1}^{\infty}\lambda^{-n}Y_n$ and the domain G is the ring $D_{a,b}$ ($a \le |\lambda| \le b$) with

the cutted piece of the real axis. The derivatives of S have now singularities both in the origin and at the infinity:

$$
\begin{aligned}
\frac{\partial S}{\partial T} &= \ln\lambda + \frac{\partial \widetilde{S}}{\partial T} \quad , \\
\frac{\partial S}{\partial X_n} &= \lambda^n + \frac{\partial \widetilde{S}}{\partial X_n} \quad , \\
\frac{\partial S}{\partial Y_n} &= \lambda^{-n} + \frac{\partial \widetilde{S}}{\partial Y_n} \qquad , \qquad n = 1,2,3,... \quad .
\end{aligned}
\tag{6.1}
$$

Since

$$
\begin{aligned}
\widetilde{S}(\lambda; X, Y, T) &= 1 + \frac{\widetilde{S}_1}{\lambda} + \frac{\widetilde{S}_2}{\lambda^2} + ... \qquad , \qquad \lambda \to \infty \quad , \\
\widetilde{S}(\lambda; X, Y, T) &= S_0 + \lambda S_1 + \lambda^2 S_2 + ... \qquad , \quad \lambda \to 0
\end{aligned}
\tag{6.2}
$$

one has at $\lambda \to \infty$

$$
\begin{aligned}
\frac{\partial S}{\partial X_n} &= \lambda^n + \frac{1}{\lambda}\frac{\partial \widetilde{S}_1}{\partial X_n} + ... \\
\frac{\partial S}{\partial Y_n} &= \lambda^{-n} + \frac{1}{\lambda}\frac{\partial \widetilde{S}_1}{\partial X_n} + ... \qquad , \qquad n = 1,2,3,... \\
e^{\frac{\partial S}{\partial T}} &= \lambda + \frac{\partial \widetilde{S}_1}{\partial T} + \frac{1}{\lambda}\left[\frac{\partial \widetilde{S}_2}{\partial T} + \frac{1}{2}\left(\frac{\partial \widetilde{S}_1}{\partial T}\right)^2\right] \qquad , \\
e^{-\frac{\partial S}{\partial T}} &= \frac{1}{\lambda} - \frac{1}{\lambda^2}\frac{\partial \widetilde{S}_1}{\partial T}
\end{aligned}
\tag{6.3}
$$

while at $\lambda \to 0$

$$
\begin{aligned}
\frac{\partial S}{\partial X_n} &= \frac{\partial S_0}{\partial X_n} + O(\lambda) \\
\frac{\partial S}{\partial Y_n} &= \frac{1}{\lambda^n} + \frac{\partial S_0}{\partial Y_n} + O(\lambda) \qquad , \qquad n = 1,2,3,... \\
e^{\frac{\partial S}{\partial T}} &= \lambda e^{\frac{\partial S_0}{\partial T}} + \lambda^2 e^{\frac{\partial S_0}{\partial T}}\frac{\partial S_1}{\partial T} + O\left(\lambda^3\right) \qquad , \\
e^{-\frac{\partial S}{\partial T}} &= \frac{1}{\lambda}e^{-\frac{\partial S_0}{\partial T}} - \frac{\partial S_1}{\partial T}e^{-\frac{\partial S_0}{\partial T}} + O(\lambda) \qquad .
\end{aligned}
\tag{6.4}
$$

The required function f_0 should not have singularities at $\lambda = 0$ and at $\lambda = \infty$ and should vanish at $\lambda \to \infty$. Taking the derivatives $\frac{\partial S}{\partial T}$, $\frac{\partial S}{\partial X_n}$, $\frac{\partial S}{\partial Y_n}$ and using

(6.2), (6.3), one finds the following two equations

$$\frac{\partial S}{\partial Y_1} - Ve^{-\frac{\partial S}{\partial T}} = 0 \qquad (6.5)$$

$$\frac{\partial S}{\partial X_1} - e^{\frac{\partial S}{\partial T}} - U = 0 \qquad (6.6)$$

where $V(X,Y,T) = e^{\frac{\partial S_0}{\partial T}}$ and $U(X,Y,T) = -\frac{\partial \widetilde{S}_1}{\partial T}$. The system (6.5)-(6.6) is the simplest system of equations for the function S associated with the d2DTL hierarchy. We note that in contrast to the papers [10] we have only one function S.

To extract from the above system nonlinear equations for the functions $V(X,Y,T)$ and $U(X,Y,T)$ we perform the expansion of the l.h.s. of equation (6.5) at large λ and of the l.h.s. of equation (6.6) around $\lambda = 0$. The terms of the order $\frac{1}{\lambda}$ in (6.5) give $V = 1 + \frac{\partial \widetilde{S}}{\partial Y_1}$ while vanishing of the zero order terms in equation (5.6) provides us with the equation $\frac{\partial S_0}{\partial X_1} - U = 0$. As a result, we get the system of equations

$$1 + \frac{\partial \widetilde{S}_1}{\partial Y_1} = e^{\frac{\partial S_0}{\partial T}} \quad , \quad \frac{\partial S_0}{\partial X_1} + \frac{\partial \widetilde{S}_1}{\partial T} = 0 \quad . \qquad (6.7)$$

To rewrite it in a more familiar form we introduce the function $\alpha = \frac{\partial S_0}{\partial T}$, the differentiate twice the first equation (6.7) with respect to T and use the second equation (6.7). One gets

$$\frac{\partial \alpha}{\partial X_1 \partial Y_1} + \frac{\partial^2}{\partial T^2}(e^{\alpha}) = 0 \qquad (6.8)$$

that is the standard form of the dispersionless 2DTL equation.

It is easy to show that the formal compatibility conditions for (6.5), (6.6) are equivalent to the system

$$V_{X_1} - V\,U_T = 0 \quad , \quad U_{Y_1} + V_T = 0 \qquad (6.9)$$

which, of course, again gives rise to equation (6.8) ($\alpha = \ln V$). In the form (6.9) the 2DTL equation has been derived in [8].

Higher equations for S can be obtained analogously. Taking the times X_2 and Y_2 , one finds the following equations

$$\frac{\partial S}{\partial Y_2} - V_2 e^{-2\frac{\partial S}{\partial T}} - V_1 e^{-\frac{\partial S}{\partial T}} = 0 \quad , \qquad (6.10)$$

$$\frac{\partial S}{\partial X_2} - e^{2\frac{\partial S}{\partial T}} - U_1 e^{\frac{\partial S}{\partial T}} = 0 \qquad (6.11)$$

where

$$\begin{array}{llll} V_2 = e^{2\frac{\partial S_0}{\partial T}} & , & V_1 = 2\,\frac{\partial S_1}{\partial T}\,e^{\frac{\partial S_0}{\partial T}} & , \\ U_1 = -2\,\frac{\partial \widetilde{S}_1}{\partial T} & , & U_0 = -2\,\frac{\partial \widetilde{S}_2}{\partial T} & . \end{array} \qquad (6.12)$$

Higher d2DTL equations have, consequently, the form

$$
\begin{aligned}
\frac{\partial V_2}{\partial X_2} - 2\,V_2\,\frac{\partial U_0}{\partial T} &= 0 \qquad , \\
\frac{\partial V_1}{\partial X_2} - V_1\frac{\partial U_0}{\partial T} - 2V_2\,\frac{\partial U_1}{\partial T} - U_1\frac{\partial V_2}{\partial T} &= 0 \qquad , \\
\frac{\partial U_0}{\partial Y_2} + \frac{\partial}{\partial T}(U_1 V_1) + 2\,\frac{\partial V_2}{\partial T} &= 0 \qquad , \\
\frac{\partial U_1}{\partial Y_2} + 2\,V_1\,\frac{\partial V_1}{\partial T} &= 0 \qquad .
\end{aligned}
\tag{6.13}
$$

The hierarchy of equations for S takes the form

$$
\frac{\partial S}{\partial Y_n} - \sum_{k=1}^{n} V_{nk}(X,Y,T)\,e^{-k\frac{\partial S}{\partial T}} = 0 \qquad , \tag{6.14}
$$

$$
\frac{\partial S}{\partial X_n} - \sum_{k=0}^{n} U_{nk}(X,Y,T)\,e^{k\frac{\partial S}{\partial T}} = 0 \tag{6.15}
$$

where V_{nk} and U_{nk} ($U_{nn} = 1$) are appropriate functions. These equations provides us with the d2DTL hierarchy for the coefficients V_{nk} and U_{nk} .

The formulae (6.14), (6.15) shows the role of the function $e^{\frac{\partial S}{\partial T}}$. In the 1+1-dimensional case this fact was first noted in the paper [5].

The d2DTL hierarchy clearly contains the dKP and dmKP hierarchies as sub-hierarchies. The first arises if one consider only times X_n putting $T_n = T = 0$ while the dmKP hierarchy is associated only with times Y_n ($X_n = T = 0$).

Equations (6.14) and (6.15) imply the hierarchy of equations for the function S only. The lowest of them is of the form

$$
\frac{\partial^2 S}{\partial X_1 \partial Y_1} + e^{\frac{\partial S}{\partial T}}\,\frac{\partial S}{\partial Y_1}\,\frac{\partial^2 S}{\partial T^2} = 0 \qquad . \tag{6.16}
$$

7 Ring of symmetries for the quasi-classical $\bar{\partial}$-problem and universal Whitham hierarchy

The results of the previous section demonstrate that the symmetries of the quasi-classical $\bar{\partial}$-problem have a rather special property. Namely, for the dKP, dmKP and d2DTL hierarchies different symmetries $\omega_A = \frac{\partial S}{\partial T_A}$ are connected by certain algebraic relations (see formulae (5.3), (5.12) and 6.14), (6.15)).

This property of the symmetries of the quasi-classical $\bar{\partial}$-problem has, in fact, a deeper background and is of general character. This background is provided by certain theorems about the solutions of the Beltrami equation (see [28]).

Thus, let us start with the general quasi-classical $\bar{\partial}$-problem

$$
\frac{\partial S}{\partial \bar{\lambda}} = W\left(\lambda, \bar{\lambda}; \frac{\partial S}{\partial \lambda}\right) \tag{7.1}
$$

where $W\left(\lambda, \bar{\lambda}; \xi\right)$ is a certain function and dependence of S on parameters (times) is not specified yet.

Infinitesimal symmetries ω of the problem (7.1) are defined by the linear Beltrami equation

$$\frac{\partial \omega}{\partial \overline{\lambda}} = W'\left(\lambda, \bar{\lambda}; \frac{\partial S}{\partial \lambda}\right) \frac{\partial \omega}{\partial \lambda} \qquad . \tag{7.2}$$

Linear Beltrami equation possesses a number of interesting properties. They have been studied in details as a part of the theory of generalized analytic functions (see [28]). The first important property is formulated in the section 3 (Chapter II) of the book [28]. This **Theorem 1** (see the appendix) states that for measurable and bounded on the entire complex plane $\mathbb{C}$ functions W' which satisfies the condition $|W'| \leq W_0 < 1$ and some other mild conditions, equation (7.2) has a solution $\omega_0(\lambda)$ (so-called, basic homeomorphism) for which

$$\omega_0(\lambda) = \lambda + O\left(\frac{1}{\lambda}\right) \qquad , \qquad \lambda \to \infty \qquad . \tag{7.3}$$

Another Theorem (the theorem 2.16 from [28])(see the appendix) says that all solutions (in some class) of equation (7.2) are given by the formula

$$\omega(\lambda, \overline{\lambda}) = \Omega\left(\omega_0(\lambda, \overline{\lambda})\right) \tag{7.4}$$

where $\Omega(\xi)$ is an arbitrary analytic function in the domain $\omega(D_0)$.

These two basic results allow us to construct infinite hierarchy associated with the problem (7.1). Indeed, let us assign the time t_A for each symmetry ω_A such that $\omega_A = \frac{\partial S}{\partial T_A}$. So for the basic solution (7.3) $\omega_0 = \frac{\partial S}{\partial T_0}$. The first theorem now says that there exists a symmetry of equation (7.1) such that

$$\frac{\partial S}{\partial T_0} = \lambda + O\left(\frac{1}{\lambda}\right) \qquad . \tag{7.5}$$

Then the second theorem states that for any symmetry $\frac{\partial S}{\partial T_A}$ one has

$$\frac{\partial S}{\partial T_A} = \Omega_A\left(\frac{\partial S}{\partial T_0}(\lambda, \overline{\lambda}), T\right) \tag{7.6}$$

where $\Omega(\xi, T)$ is an appropriate function of the first argument. Thus the above two theorems imply that under certain conditions the $\bar{\partial}$-equation (7.1) possesses an infinite ring of symmetries (deformations) given by the equations

$$\frac{\partial S(\lambda, \overline{\lambda}; T)}{\partial T_A} = \Omega_A\left(\frac{\partial S}{\partial T_0}, T\right) \qquad , \qquad A = 0, 1, 2, 3, ... \tag{7.7}$$

where $\Omega_A(\xi, T)$ are arbitrary analytic functions of the ξ. The set of the equations (7.7) is compatible by construction. Equations (7.7) give rise to certain nonlinear

equations for functions $U_k(T)$ on which S may depend. These equations can be obtained also from the equations for Ω_A which follow from (7.7). They are

$$\frac{\partial \Omega_A}{\partial T_B} - \frac{\partial \Omega_B}{\partial T_A} + \{\Omega_A, \Omega_B\} = 0 \qquad , \qquad A, B = 0, 1, 2, ... \tag{7.8}$$

where

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial T_1} - \frac{\partial f}{\partial T_1}\frac{\partial g}{\partial p} \qquad . \tag{7.9}$$

and we denoted $p = \frac{\partial S}{\partial T_0}$.

So we constructed an integrable hierarchy of equations out of the $\bar{\partial}$-problem (7.1). It is an infinite ring since $\Omega_A(\xi, T)$ are arbitrary analytic functions. To get a concrete hierarchy one has to specify the set of functions Ω_A. The set of functions $\Omega_A\left(p(\lambda, \overline{\lambda}; T), T\right)$ such that $\Omega_k \sim \lambda^k + O\left(1/\lambda\right)$ as $\lambda \to \infty$ with identification $T_0 = T_1$, $T_k = T_{A-1}$ gives rise to the dKP hierarchy.

In the construction given above the time T_0 has played a special role being connected with the "basic" symmetry ω_0. Infinite ring of symmetries for the problem (7.1) admits more general and symmetric formulation. It is due to the already mentioned obvious fact that any differentiable function $f\left(\frac{\partial S}{\partial T_{A_1}}, \frac{\partial S}{\partial T_{A_2}}, ...\right)$ of any set of symmetries $\frac{\partial S}{\partial T_{A_1}}, \frac{\partial S}{\partial T_{A_2}}$, ... is again a symmetry (*i.e.* a solution of equation (7.2)). Then the implicit function theorem implies that any symmetry $\frac{\partial S}{\partial T_A}$ can be chosen as a basic one.

So, let us take (arbitrary) symmetry $p = \frac{\partial S}{\partial T_0}$. The infinite hierarchy of symmetries now take the form

$$\frac{\partial S}{\partial T_A} = \Omega_A(p, T) \tag{7.10}$$

where $\Omega_A(\xi, T)$ are arbitrary differentiable functions of ξ. The compatibility conditions for equation (7.10) is of the form (7.8), (7.9) where now T_0 and $p = \frac{\partial S}{\partial T_0}$ are arbitrary time and the corresponding symmetry.

The infinite set of equations (7.8) in this case is nothing but the universal Whitham hierarchy introduced in the different way in [8]. So in our approach the universal Whitham hierarchy is an infinite ring of symmetries of the general quasi-classical $\bar{\partial}$-problem (7.1).

8 Dispersionless hierarchies of the B type

Various type of reductions for the dKP hierarchy have been considered in [5], [10]. Here we will discuss the dispersionless hierarchies of the so-called B type. The dispersionless BKP hierarchy has been discussed briefly in the paper [30]. The dBKP hierarchy is characterized by the constraint [30]

$$S(-\lambda, T) = -S(\lambda, T) \qquad . \tag{8.1}$$

This constraint immediately implies that only odd powers of $\frac{\partial S}{\partial T_1}$ are allowed in the equations (5.3). Since in this case $S_0(\lambda,T)=\lambda\, T_1+\lambda^3 T_3+\lambda^5 T_5+...$ and $\widetilde{S}=1+\frac{S_1}{\lambda}+\frac{S_3}{\lambda^3}+...$ as $\lambda\to\infty$, the hierarchy of equations for S takes the form

$$\frac{\partial S}{\partial T_{2n+1}}-\left(\frac{\partial S}{\partial T_1}\right)^{2n+1}-\sum_{k=0}^{n-1}U_{nk}(T)\,\left(\frac{\partial S}{\partial T_1}\right)^{2k+1}=0 \quad . \tag{8.2}$$

The two lowest equations (8.2) are

$$\frac{\partial S}{\partial T_3}-\left(\frac{\partial S}{\partial T_1}\right)^3-U\,\frac{\partial S}{\partial T_1}=0 \quad , \tag{8.3}$$

$$\frac{\partial S}{\partial T_5}-\left(\frac{\partial S}{\partial T_1}\right)^5-V_3\,\left(\frac{\partial S}{\partial T_1}\right)^3-V_1\,\frac{\partial S}{\partial T_1}=0 \tag{8.4}$$

where

$$U=-3\frac{\partial S_1}{\partial T_1}\quad,\quad V_3=\frac{5}{3}U\quad,\quad V_1=\frac{5}{9}U^2-\frac{\partial S_3}{\partial T_1}\quad . \tag{8.5}$$

Equations (8.3), (8.4) implies that

$$\frac{9}{5}\frac{\partial U}{\partial T_5}+U^2\frac{\partial U}{\partial T_1}-U\frac{\partial U}{\partial T_3}-\frac{\partial U}{\partial T_1}\,\partial_{T_1}^{-1}\left(\frac{\partial U}{\partial T_3}\right)-\partial_{T_1}^{-1}\left(\frac{\partial^2 U}{\partial T_3^2}\right)=0 \tag{8.6}$$

Equation (8.5) is the dispersionless limit of the 2+1-dimensional Sawada-Kotera (and also Kaup-Kupershmidt) equation [31],[32].

To get the d2DTL hierarchy of the B type we shall use the universal Whitham hierarchy equation (7.10) and (7.7). Due to constraint (8.1) only odd functions $\Omega_A(-p,T)=-\Omega(p,T)$ are admissable. Taking the time $t_1=X$ as the reference one ($i.e.$ $p=\frac{\partial S}{\partial X}$) and two other equations (7.10) in the form

$$\frac{\partial S}{\partial Y}=\frac{V}{p-U}-\frac{V}{p+U}\qquad,\qquad\frac{\partial S}{\partial T}=\ln\frac{p-U}{p+U} \tag{8.7}$$

where u and V are functions of X,Y,T , one obtains the equations

$$\begin{aligned}&V_T+U_Y=0 \qquad ,\\ &U_T+\tfrac{U_X}{U}-\tfrac{V_X}{V}=0 \qquad .\end{aligned} \tag{8.8}$$

Introducing the function $\beta=\ln\left(\frac{V}{U}\right)$, one can rewrite the system (8.8) as

$$\begin{aligned}&\beta_{XY}+\left(Ue^{\beta}\right)_{TT}=0 \qquad ,\\ &\beta_X+U_T=0 \qquad .\end{aligned} \tag{8.9}$$

It is the d2DTL equation of the B type. The analog of equation (6.5), 6.6) for the B-d2DTL equation (8.9) is rather interesting

$$\begin{aligned} &\tfrac{\partial S}{\partial Y} + \tfrac{V}{U}\, sh\left(\tfrac{\partial S}{\partial T}\right) = 0 \qquad , \\ &\tfrac{\partial S}{\partial X} + U\, cth\left(\tfrac{1}{2}\tfrac{\partial S}{\partial T}\right) = 0 \qquad . \end{aligned} \tag{8.10}$$

The compatibility condition for this system is equivalent to the system (8.8).

Note finally that the nonlinear equation for $S(z, \bar{z}; X, Y, T)$ in this case is of the form

$$\frac{S_{TT}\, S_X\, S_Y}{1 + ch(S_T)} - \frac{S_{TX}\, S_Y}{sh(S_T)} + S_{XY} = 0 \qquad . \tag{8.11}$$

APPENDIX

There is a well established theory of generalized solutions of the linear Beltrami equation (see for instance [21]-[23] and [28])

$$Z_{\bar{\lambda}} = A Z_{\lambda}, \tag{A.1}$$

where A is any given measurable function $||A||_\infty < 1$ on G. Obviously, for $A \equiv 0$ we get into the class of conformal mappings. To present these results we need to introduce the operators

$$Th(\lambda) := \frac{1}{2\pi\imath}\iint_{\mathbb{C}\times\mathbb{C}} \frac{h(\lambda')}{\lambda' - \lambda}\mathrm{d}\lambda' \wedge \bar{\lambda}', \quad \Pi h(\lambda) := \frac{\partial Th}{\partial \lambda}(\lambda),$$

where the integral is taken in the sense of the Cauchy principal value. Then one has:

Lemma. *For any $p > 1$ the operator Π defines a bounded operator in $L^p(\mathbb{C})$ and for any $0 \le k < 1$ there exists $\delta > 0$ such that*

$$k||\Pi||_p < 1,$$

for all $|p - 2| < \delta$.

The next theorem summarizes the properties of solutions of (A.1) that we need in our discussion.

Theorem. *Given a measurable function A with compact support inside the circle $|\lambda| < R$ and such that $||A||_\infty < k < 1$. Then, for any fixed exponent $p = p(k) > 2$ such that $k||\Pi||_p < 1$, it follows that*

1) *There is a unique function Z_0 on $\mathbb{C}$ with distributional derivatives satisfying the Beltrami equation (A.1) such that*

$$Z_0(\lambda) = \lambda + O(\frac{1}{\lambda}), \quad \lambda \to \infty, \tag{A.2}$$

with $Z_{0,\bar{\lambda}}$ and $Z_{0,\lambda} - 1$ being elements of $L^p(\mathbb{C})$.

2) *Every solution of* (A.2) *on a domain* G *of* $\mathbb{C}$ *can be represented as*

$$Z(\lambda) = \Phi(Z_0(\lambda)), \tag{A.3}$$

where Φ *is an arbitrary analytic function on the image domain* $Z_0(G)$ *of* G *under* Z_0.

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